Math 821, Spring 2013

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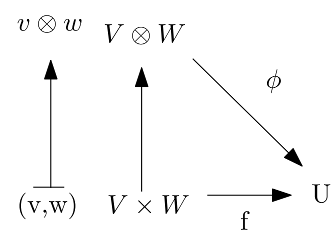
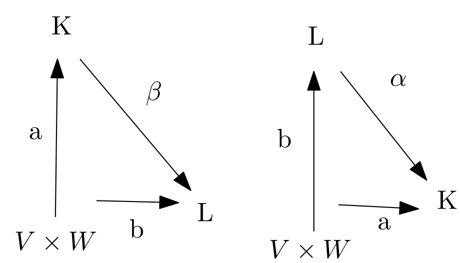
Algebras

Definition. Let k be a field. An (associated) **algebra** A over k is a vector space with a binary operation $\cdot$ satisfying
(1) $\cdot$ is associative
(2) $\cdot$ distributes over $+$ on the left and right
(3) $\cdot$ is compatible with scalar multiplication. i.e. $a(\lambda b) = \lambda(ab)\ \forall \lambda \in k$
(4) A has a multiplicative unit. i.e. $\exists \mathbf{1} \in A$ s.t. $\mathbf{1}a = a = a\mathbf{1}$.
Lets say this more algebraically.

Definition. Let V and W be k-vector spaces. Then there is a unique pair of a vector space $V \otimes W$ and a bilinear map $V \times W \to V \otimes W$ satisfying the following property.
$\forall$ bilinear maps $f : V \times W \to U\ \exists! \phi : V \otimes W \to U$ s.t. the following diagram commutes.



Note
1.) The tensor produce is a machine for turning bilinear maps into linear maps.
2.) Whenever something satisfies a universal property like that it is unique if it exists. Suppose K and L both satisfied the property then they each have $V \times W \xrightarrow{a} K$ and $V \times W \xrightarrow{b} L$ and



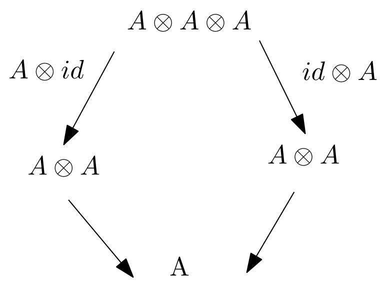
so α and β are isomorphisms.
3.) It would remain to show that the tensor product exists. (for details look at your algebra notes)

Take the elements of $V \times W$ and build the vector space generated by these elements. Then mod out by

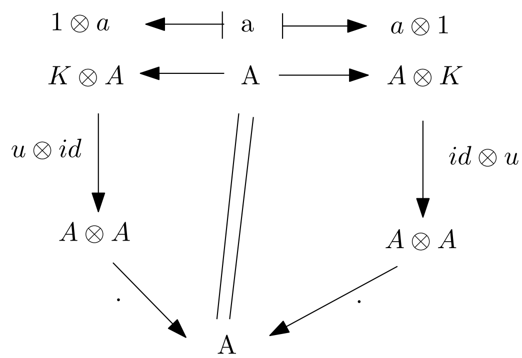
$$(a_1 + a_2) \otimes b - a_1 \otimes b - a_2 \otimes b$$
$$a \otimes (b_1 + b_2) - a \otimes b_1 - a \otimes b_2$$
$$\lambda a \otimes b - a \otimes \lambda b$$
$$\lambda a \otimes b - \lambda(a \otimes b)$$

and everything it generates. Then check this works.
With this we can rewrite the definition of an algebra with $\cdot$ linear from $A \otimes A$ to A instead of bilinear from $A \times A$ to A. This takes care of (2) and (3). What about (1). Write it as a commutative diagram.

(ab)c=a(bc)



commutes.
What about (4)?
The unit tells us how to put K inside A, namely $K \ni 1 \mapsto \mathbf{1} \in A$ so $K \ni \lambda \mapsto \lambda\mathbf{1} \in A$. So the unit gives a linear map $k \to A$ call it u. Then the unit property is that

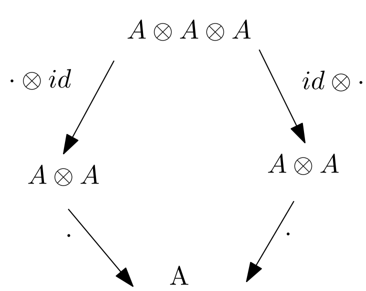


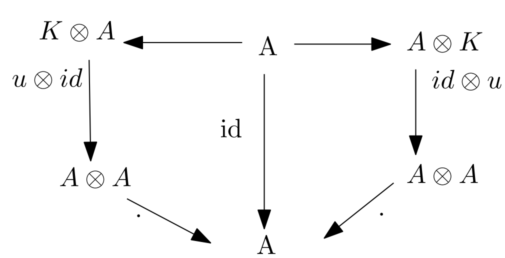
commutes.

Definition. An algebra A over K is a vector space over K with two linear maps

$$\cdot : A \otimes A \to A$$
$$u : K \to A$$

satisfying





commute.

Example. Let A be a K vector space.

Definition. (Tensor Algebra) $TA = \bigoplus_{i=0}^{\infty} A^{\otimes i}$ where $A^{\otimes 0} = K$ and $A^{\otimes i} = A \otimes A \otimes \cdots \otimes A$ with A appearing i times. This is a vector space by definition and we can make it an algebra via $(a_1 \otimes \cdots \otimes a_i)(b_1 \otimes \cdots \otimes b_j) = a_1 \otimes \cdots \otimes a_i \otimes b_1 \otimes \cdots \otimes b_j$. This is called the tensor algebra. We can view it as an algebra of words. Elements are formal linear combination of words viewing $a_1 \otimes a_2 \otimes \cdots \otimes a_i = a_1 a_2 \cdots a_i$ a word with letters from the alphabet A and where the product of 2 words is their concatenation. Note if we take TA and mod out by the action of the symmetric group or equivalently by the ideal of commutators then we get SA the symmetric algebra which we can view as an algebra of multisets with formal + and union as the product.

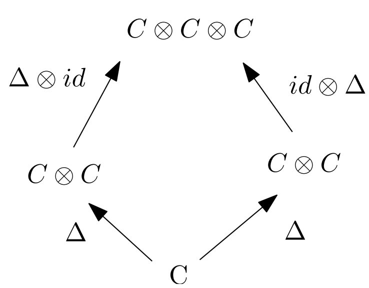
Coalgebras and Bialgebras

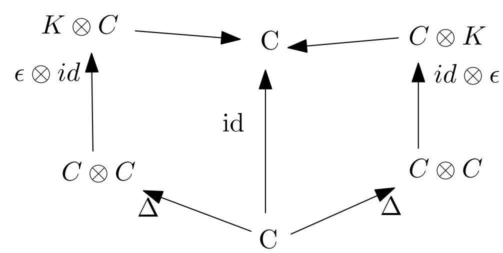
Multiplication tells you how to take 2 things and put them together to make 1 thing. Comultiplication is the opposite, it tells you how to take one thing and pull it apart into 2 things. (possibly in more than one way)

Definition. A **coalgebra** C over K is a vector space over K with 2 linear maps

$$\Delta : C \to C \otimes C$$
$$\epsilon : C \to K$$

satisfying

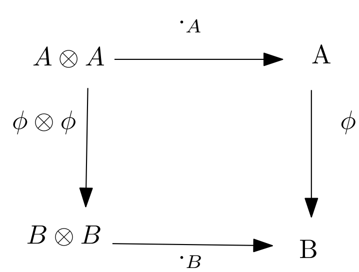


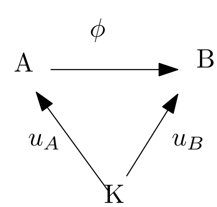


commute.

What does it mean to be an algebra homomorphism in commutative diagram language?

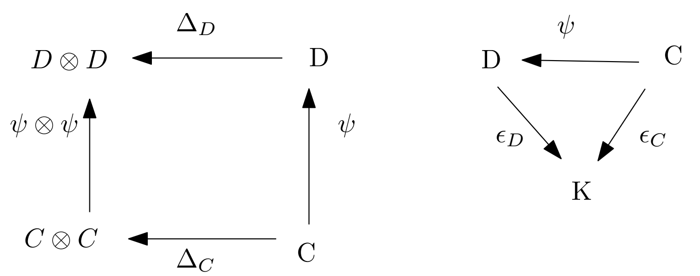
Definition. Let A and B be $K-$algebras and $\phi : A \to B$ is an algebra homomorphism if it is linear and $\phi(a_1 a_2) = \phi(a_1)\phi(a_2)$, $\phi(\mathbf{1}_A) = \mathbf{1}_B$.



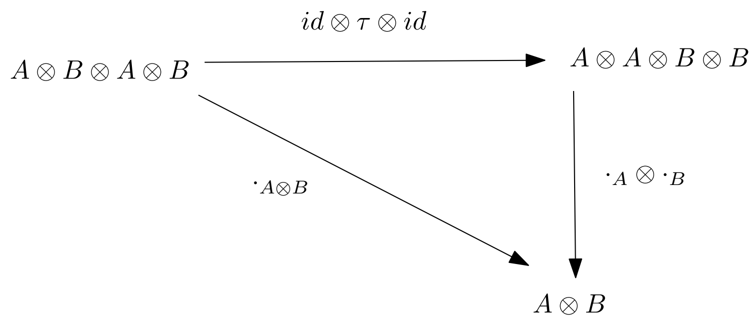


commute.

Definition. Let C and D be $K-$coalgebras. Then $\psi : C \to D$ is a coalgebra morphism if it is linear and the reverse diagrams commute. i.e.



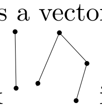
Definition. Let A and B be K algebras, then $A \otimes B$ is a $K-$algebra with the product $(a \otimes b)(c \otimes d) = ac \otimes bd$. i.e.

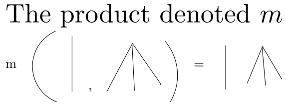


where $\tau(a \otimes b) = b \otimes a$ (transposition).
Note: not the same as the multiplication in the tensor algebra. So a bialgebra is an algebra and a coalgebra which play well together namely

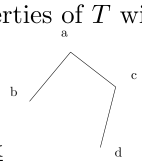
Definition. Let A be a $K-$vector space with an algebra structure $(\cdot, u)$ and a coalgebra structure (Δ, ϵ). If Δ and ϵ are algebra homomorphisms then A with $(\cdot, u, \Delta, \epsilon)$ is a **bialgebra**.

Example. The Connes-Kreimer Hopf Algebra (just bialgebras today) of rootes trees.
As a vector space a basis is the set of forest of rooted trees i.e. MSets of rooted trees.

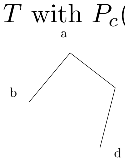
ex  is a basis element.
The product denoted m is disjoint union



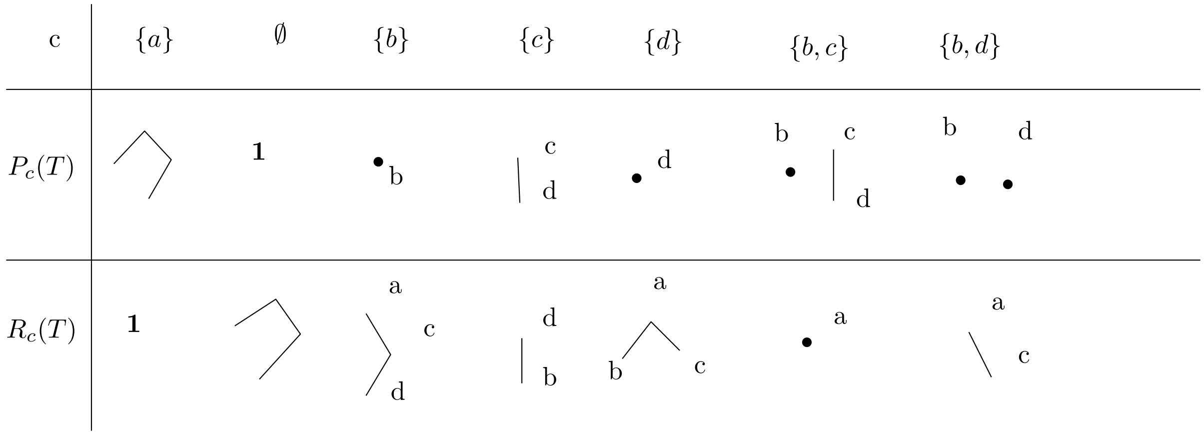
and so another way to think of this is as the polynomial algebra generated by rooted trees. As an element the unit is the empty tree. Write it $\mathbf{1}$ (not ϵ to avoid confusion). What about the coproduct. We need a definition. Given a rooted tree T an admissible cut of T is a set possibly empty, of verties of T with the property that no vertex in the set is a descendant of another.

ex 

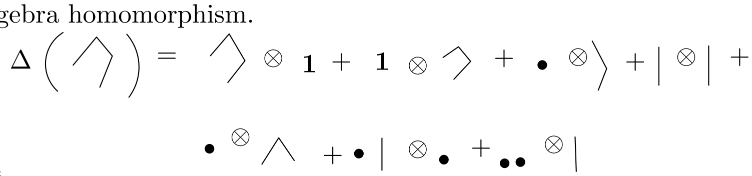
admissible cuts $= \{\{a\}, \emptyset, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}\}$. Now for T a rooted tree, c an admissible cut of T, define $P_c(T)$ to be the forest of subtrees of T which are rooted at elements of c and define $R_c(T)$ to be T with $P_c(T)$ removed. P for "pruned" and R for "root".

ex for 

again we get



Then the coproduct is $\Delta(T) = \sum_{c \text{ admissible cut}} P_c(T) \otimes R_c(T)$ extend to all elements of the bialgebra as an algebra homomorphism.



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References. V Reiner's notes "Hopf Algebras in Combinatorics" 1.1, 1.2, 1.3.